

# The SUMPLE Algorithm for Aligning Arrays of Receiving Radio Antennas: Coherence Achieved with Less Hardware and Lower Combining Loss

D. H. Rogstad<sup>1</sup>

*Analysis and simulations are presented to show that coherence between the set of receiving antennas that form an array can be achieved with hardware and processing that are proportional to the number of antennas rather than the square of the number of antennas.*

## I. Introduction

Receiving-antenna arraying is quickly becoming a very important capability within the Deep Space Network (DSN). Instead of using a single large antenna to receive telemetry signals from distant spacecraft, several smaller antennas are used in concert. The signal received at each small antenna is adjusted in delay and phase to bring it into coherence with the corresponding signals received by the other antennas. These adjusted signals add to yield a combined signal having improved signal-to-noise ratio (SNR).

Although antenna arraying has been employed by the DSN to enhance the data return from several space missions in the past [1], it was used in a major way to assist in “saving” the Galileo Mission [2], when that spacecraft’s large deployable antenna failed to open, severely limiting its telemetry rate from Jupiter. Combining the output from several large DSN ground-based antennas provided sufficient boost in SNR to allow greater than 10 times improvement in Galileo’s reduced telemetry rate. The approach was considered so successful that a follow-on, more general-purpose array capable of combining up to six 34-meter antennas was implemented, and the DSN is now in the process of designing a very large array with 400 12-meter antennas. This last instrument will have the equivalent receiving capability of up to 40 times that of a single 70-meter antenna.

There are at least three key technologies that have either enabled or greatly enhanced the capability of these arrays: the full-spectrum arraying approach [3]; the fractional-clock sample delay technology [4]; and, as described here, an array coherence technique called “SUMPLE.” In achieving coherence among the various antennas, SUMPLE reduces the amount of needed signal-processing hardware by a factor of  $(N - 1)/2$  (where  $N$  is the number of antennas in the array) while at the same time providing very

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<sup>1</sup> Tracking Systems and Applications Section.

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low combining loss for weak signals. The name SUMPLE is derived from its predecessor, the SIMPLE algorithm, in which array coherence is achieved by the use of a simple pair-wise correlation of the antenna signals. In contrast, SUMPLE sums a number of antenna signals before the correlation operation is performed.

## II. Description

The SUMPLE algorithm was first proposed by the author in a large-array study performed in 1994 [5] but was not extensively investigated until the implementation of the DSN 34-meter array [6]. The technique is diagrammed in Fig. 1. Instead of correlating each antenna signal of the array with a reference-antenna signal in a pair-wise fashion, the SUMPLE algorithm correlates each antenna signal with the sum of the other antenna signals.

This has two benefits. First, it reduces the correlation hardware from  $(N) * (N - 1)/2$  correlators down to  $N$  correlators as well as reduces the post-correlation data handling and processing by a similar factor. These provide a very significant hardware cost reduction for large arrays. Second, each correlation shows a factor of  $(N - 1)$  improvement in SNR once the array achieves coherence. This is critical in the weak-signal environment where arraying is especially useful.

Simulation of SUMPLE along with several other algorithms has been performed (see [7] for some discussion of these other algorithms). Results, shown for a 25-antenna array in Fig. 2, indicate that, after array lock has been achieved, all of the algorithms have greater combining loss for lower correlation averaging times (equivalent to larger algorithm bandwidth) but improve as this averaging time increases (algorithm bandwidth narrows). For antenna baselines (distance between antennas) greater than about a kilometer, the correlation averaging time is ultimately limited to a few seconds for telemetry transmitted at Ka-band (32-GHz range) due to tropospheric phase instabilities causing de-correlation and incoherence between antennas. For the algorithms that require only  $(N - 1)$  or  $N$  correlators (SIMPLE and SUMPLE), SUMPLE performs best. For the others that require  $(N) * (N - 1)/2$  correlators (LSFIT and EIGEN), LSFIT performs best, but is only slightly better than SUMPLE. These and further simulation results are presented in Section VII.

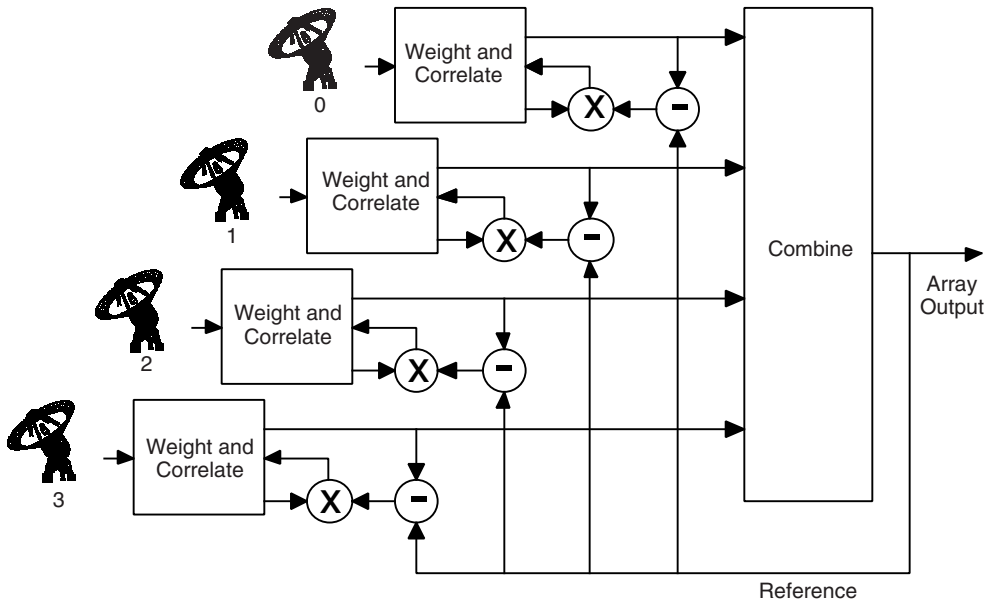
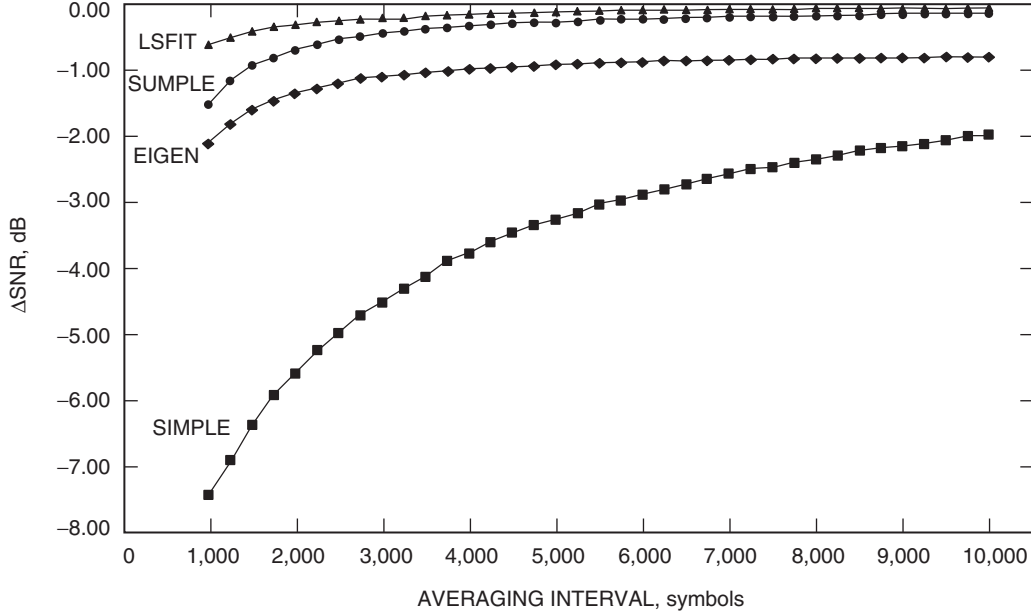


Fig. 1. Diagram for the SUMPLE method of combining.



**Fig. 2. Chart of combining loss for four different correlation algorithms. Longer integration is equivalent to narrower algorithm bandwidth.**

In the analysis and simulations presented in this study, it has been assumed that symbol synchronization between the various antenna signals has been accomplished. If this is not the case, the correlation SNRs are reduced by about 1.5 dB. When necessary, this degradation can be overcome by increasing the correlation averaging interval by a corresponding factor.

### III. Calculation of the Combined Signal plus Noise

#### A. Non-Normalized Weight

For the purpose of simulating the processing necessary to achieve array coherence, the signal from the  $i$ th antenna can be represented by

$$\hat{S}_{ik} = [\hat{s}_{ik} + \hat{n}_{ik}^s] \quad (1)$$

where  $k$  is its time index. In the case of telemetry signals, there is no loss of generality by assuming that integration over a symbol has already occurred. The signal in this expression is represented by  $\hat{s}_{ik}$ , and the noise is represented by  $\hat{n}_{ik}^s$ . The hat ( $\hat{\phantom{x}}$ ) over the various quantities indicates complex notation. This signal is defined at the input to the signal-processing equipment of the combiner and includes the phase effects of the troposphere and the instrument, as discussed later.

The weight used to obtain a weighted sum of the various antennas is represented by

$$\hat{W}_{iK} = [\hat{\omega}_{iK} + \hat{\eta}_{iK}] \quad (2)$$

where  $K$  is the time index in units of the correlation averaging interval,  $ncor$ . This weight will be derived, in Section IV, from averaging the cross-correlations of the antennas with each other. Therefore, the noise,  $\hat{\eta}_{iK}$ , on the weight,  $\hat{\omega}_{iK}$ , comes from the correlation noise, and is reduced by this correlation averaging.

The combined output of the array is then

$$\hat{C}_k = \sum_{i=0}^{N_{ant}-1} \hat{S}_{ik} \hat{W}_{iK}^* \quad (3)$$

where the sum is performed over the total number of antennas,  $N_{ant}$ , and the  $*$  represents complex conjugation. Plugging in the expressions for signal and weight yields

$$\hat{C}_k = \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^* + \hat{s}_{ik} \hat{\eta}_{iK}^* + \hat{n}_{ik}^s \hat{\omega}_{iK}^* + \hat{n}_{ik}^s \hat{\eta}_{iK}^*) \quad (4)$$

If we let the combined output be represented by

$$\hat{C}_k = [\hat{c}_k + \hat{n}_k^c] \quad (5)$$

the combined signal and noise components become

$$\hat{c}_k = \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^* + \hat{s}_{ik} \hat{\eta}_{iK}^*) \quad (6)$$

$$\hat{n}_k^c = \sum_{i=0}^{N_{ant}-1} (\hat{n}_{ik}^s \hat{\omega}_{iK}^* + \hat{n}_{ik}^s \hat{\eta}_{iK}^*) \quad (7)$$

Assuming all the antennas are equal in size and performance, and the array to be “in lock” (i.e., all the signals, within the noise, are adding coherently), the combined signal power is

$$|\hat{c}_k|^2 = \left| \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^*) + \sqrt{N_{ant}} |\hat{s}_k| |\hat{\eta}_K^*| \right|^2 \quad (8)$$

The second term within the vertical bars of this expression has been derived from Eq. (6) assuming the weight noise between the various antennas is uncorrelated. While this is not strictly true (because the noise of the reference antenna is common to all the weights in a given time interval), it is true to a good approximation if the individual antenna signal-to-noise ratios are somewhat less than one, or if the uncorrected antenna signal phases are fairly random at start-up. Both of these conditions are often true in a large array. After expanding, this expression becomes

$$|\hat{c}_k|^2 = \left| \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^*) \right|^2 + N_{ant} |\hat{s}_k|^2 |\hat{\eta}_K|^2 \quad (9)$$

For these and all following power expressions, we assume that they have been time averaged without being explicitly shown.

Under the same assumptions, the noise power is

$$|\hat{n}_k^c|^2 = \left| \sqrt{N_{ant}} (\hat{n}_k^s \hat{\omega}_K^* + \hat{n}_k^s \hat{\eta}_K^*) \right|^2 \quad (10)$$

which becomes

$$|\hat{n}_k^c|^2 = N_{ant} \left[ |\hat{\omega}_K|^2 |\hat{n}_k^s|^2 + |\hat{n}_k^s|^2 |\hat{\eta}_K|^2 \right] \quad (11)$$

Using Eqs. (9) and (11) for combined signal and noise, we get a combining signal-to-noise ratio of

$$\text{SNR} = \frac{|\hat{c}_k|^2}{|\hat{n}_k^c|^2} = \frac{\frac{1}{N_{ant}} \left| \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^*) \right|^2 + |\hat{s}_k|^2 |\hat{\eta}_K|^2}{\left[ |\hat{\omega}_K|^2 |\hat{n}_k^s|^2 + |\hat{n}_k^s|^2 |\hat{\eta}_K|^2 \right]} \quad (12)$$

which, with some manipulation, yields

$$\text{SNR} = \frac{\frac{1}{N_{ant} |\hat{s}_k|^2 |\hat{\omega}_K|^2} \left| \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^*) \right|^2 + \frac{|\hat{\eta}_K|^2}{|\hat{\omega}_K|^2}}{\left[ \frac{|\hat{n}_k^s|^2}{|\hat{s}_k|^2} \left( 1 + \frac{|\hat{\eta}_K|^2}{|\hat{\omega}_K|^2} \right) \right]} \quad (13)$$

If we define the ratio  $\rho$  (which we will see later is  $\approx 1$ ) as

$$\rho = \frac{1}{N_{ant}^2 |\hat{s}_k|^2 |\hat{\omega}_K|^2} \left| \sum_{i=0}^{N_{ant}-1} (\hat{s}_{ik} \hat{\omega}_{iK}^*) \right|^2 \quad (14)$$

and let the individual antenna signal SNR be  $\alpha = |\hat{s}_k|^2 / |\hat{n}_k^s|^2$  and the weight SNR be  $\gamma = |\hat{\omega}_K|^2 / |\hat{\eta}_K|^2$  (again, assuming equal antennas), then the combining SNR becomes

$$\text{SNR} = N_{ant} \alpha \left[ \frac{\gamma \rho + \frac{1}{N_{ant}}}{\gamma + 1} \right] \quad (15)$$

When  $\gamma \gg 1$ , the combining SNR becomes, as expected,

$$\text{SNR}_0 = N_{ant} \alpha \quad (16)$$

We can define a correction factor for the SNR as

$$\Delta \text{SNR}(\gamma) = \left[ \frac{\gamma \rho + \frac{1}{N_{ant}}}{\gamma + 1} \right] \quad (17)$$

The combining SNR then can be written as

$$\text{SNR} = \text{SNR}_0 \Delta \text{SNR}(\gamma) \quad (18)$$

## B. Normalized Weight

A slightly improved combining SNR can be obtained if the individual weights are normalized as represented in the expression

$$\hat{W}_{iK}^N = \frac{[\hat{\omega}_{iK} + \hat{\eta}_{iK}]}{\sqrt{|\hat{\omega}_{iK}|^2 + |\hat{\eta}_{iK}|^2}} = [\hat{\omega}_{iK}^{N*} + \hat{\eta}_{iK}^{N*}] \quad (19)$$

With this modification of the weights, it is convenient to perform some of the combining operations in polar coordinates for deriving the combining SNR. Following the procedure used above, the combined output of the array is

$$\hat{C}_k = \sum_{i=0}^{N_{ant}-1} \left( \hat{s}_{ik} \hat{W}_{iK}^{N*} + \hat{n}_{ik}^s \hat{W}_{iK}^{N*} \right) \quad (20)$$

and the signal and noise components are

$$\hat{c}_k = \sum_{i=0}^{N_{ant}-1} \left( \hat{s}_{ik} \hat{W}_{iK}^{N*} \right) \quad (21)$$

$$\hat{n}_k^c = \sum_{i=0}^{N_{ant}-1} \left( \hat{n}_{ik}^s \hat{W}_{iK}^{N*} \right) \quad (22)$$

To evaluate these expressions, we remember that  $\hat{W}_K^N$  has fixed amplitude of unity with a Gaussian-like distribution in phase around some average fixed phase. The phase probability density for  $\hat{W}_K^N$  is [8]

$$p(\phi, \gamma) = \frac{1}{2\pi} e^{-\gamma} \left[ 1 + \sqrt{2\gamma} \cos(\phi) e^{\gamma \cos^2(\phi)} \int_{-\infty}^{\sqrt{2\gamma} \cos(\phi)} e^{-x^2/2} dx \right] \quad (23)$$

where  $\phi$  is the difference between the actual phase and its average value. For the case of the array being in lock and having identical antennas, we get the following for the average of the combined signal power:

$$|\hat{c}_k|^2 = N_{ant}^2 |\hat{s}_k|^2 \left| \int_{-\pi}^{\pi} p(\phi, \gamma) \cos(\phi) d\phi \right|^2 \quad (24)$$

and the average of the combined noise power:

$$|\hat{n}_k^c|^2 = N_{ant} |\hat{n}_k^s|^2 \quad (25)$$

Using these two expressions, we obtain a combining signal-to-noise ratio of

$$\text{SNR} = \frac{|\hat{c}_k|^2}{|\hat{n}_k^c|^2} = \frac{N_{ant}^2 |\hat{s}_k|^2 \left| \int_{-\pi}^{\pi} p(\phi, \gamma) \cos(\phi) d\phi \right|^2}{N_{ant} |\hat{n}_k^s|^2} \quad (26)$$

With some further manipulation, where we use the definitions for  $\alpha = |\hat{s}_k|^2 / |\hat{n}_k^s|^2$  and  $\gamma = |\hat{\omega}_K|^2 / |\hat{\eta}_K|^2$  as above, this becomes

$$\text{SNR} = N_{ant} \alpha \Delta \text{SNR}(\gamma) \quad (27)$$

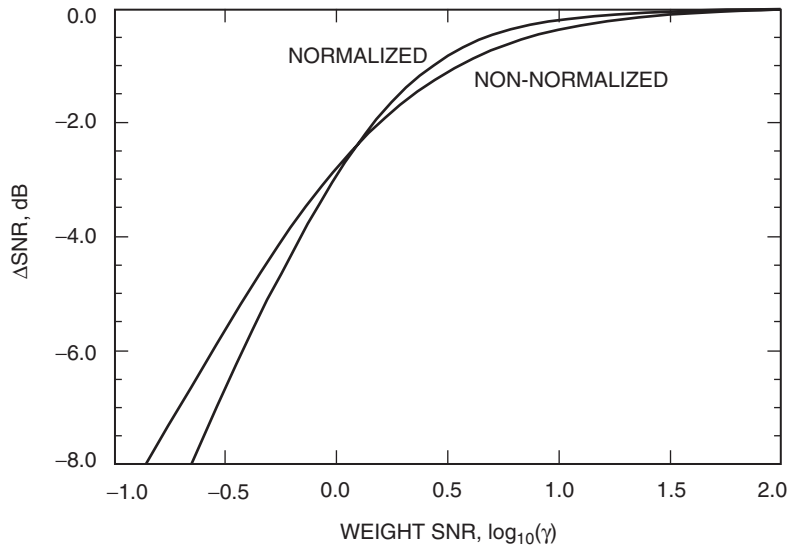
where

$$\Delta \text{SNR}(\gamma) = \left| \int_{-\pi}^{\pi} p(\phi, \gamma) \cos(\phi) d\phi \right|^2 \quad (28)$$

Following the procedure used for the non-normalized weight, the new signal-to-noise ratio becomes

$$\text{SNR} = \text{SNR}_0 \Delta \text{SNR}(\gamma) \quad (29)$$

The  $\Delta_{\text{SNR}}$  expression in Eq. (28) has been numerically integrated and is plotted in Fig. 3 together with the non-normalized result from Section III.A, Eq. (17). The normalized curve is better except for weight SNRs below about 1 (i.e.,  $\log_{10}(\gamma) < 0$ ), not surprising since normalization is derived from division by a noisy amplitude.



**Fig. 3. Combining SNR correction factors for normalized and non-normalized weights.**

#### IV. Calculation of the SIMPLE Weight

For comparison of analysis as well as simulation, we first calculate the weight for SIMPLE. Using the same expressions for signal and weight defined previously in Section III,

$$\hat{S}_{ik} = [\hat{s}_{ik} + \hat{n}_{ik}^s] \quad (30)$$

and

$$\hat{W}_{i0K} = [\hat{\omega}_{i0K} + \hat{\eta}_{i0K}] \quad (31)$$

where we use the subscript “ $i0$ ” to explicitly show that the weight is derived from a correlation of the  $i$ th antenna and the 0th, or reference, antenna, we get the following for the SIMPLE weight in the next, or  $(K + 1)$ th, time interval:

$$\hat{W}_{i0K+1} = R_{K+1} \hat{W}_{i0K} \left\{ \frac{1}{ncor} \sum_{k=Kncor}^{(K+1)ncor-1} \left[ \left( \hat{W}_{i0K}^* \hat{S}_{ik} \right) \hat{S}_{0k}^* \right] \right\} \quad (32)$$

where the sum of  $ncor$  time points (which here is  $ncor$  symbols because of the assumption of having integrated over symbols) corresponds to the correlation averaging interval. The expression within the curly brackets is an incremental weight, which when combined with the original weight used in the interval yields the weight for the next time interval. The factor  $R_{K+1}$  provides normalization to guarantee the sum of the weights-squared over the antennas is equal to the number of antennas and prevents successive calculations of the weight amplitude from becoming unstable:

$$\sum_{i=0}^{N_{ant}-1} \hat{W}_{i0K} \hat{W}_{i0K}^* = N_{ant} \quad (33)$$

In order to better visualize the various intervals contained in Eq. (32) for  $\hat{W}_{i0K+1}$  above, refer to the diagram in Fig. 4.

Expanding  $\hat{W}_{i0K+1}$  using the above definitions for signal and weight, the weight becomes

$$\hat{W}_{i0K+1} = R_{K+1} \hat{W}_{i0K} \frac{1}{ncor} \sum_{k=Kncor}^{(K+1)ncor-1} \hat{W}_{i0K}^* [\hat{s}_{ik} \hat{s}_{0k}^* + \hat{s}_{ik} \hat{n}_{0k}^{s*} + \hat{n}_{ik}^s \hat{s}_{0k}^* + \hat{n}_{ik}^s \hat{n}_{0k}^{s*}] \quad (34)$$

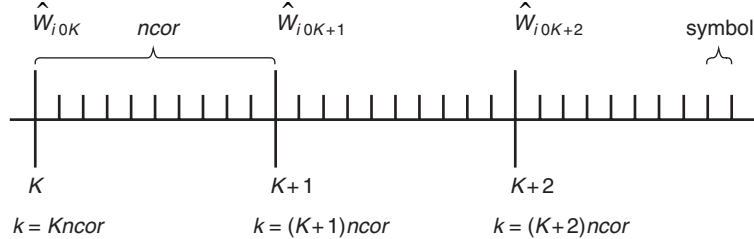


Fig. 4. Time (in symbols) and integration intervals.



Assuming a relatively constant signal and noise power over the  $ncor$  summation (corresponding to integration, or averaging, in time) allows us to perform that sum, to give

$$\hat{W}_{i0K+1} = R_{K+1} \left| \hat{W}_{i0K} \right|^2 \begin{bmatrix} \hat{s}_{iK} \hat{s}_{0K}^* \\ + \frac{1}{\sqrt{ncor}} \hat{s}_{iK} \hat{n}_{00}^{s*} \\ + \frac{1}{\sqrt{ncor}} \hat{n}_{i0}^s \hat{s}_{0K}^* \\ + \frac{1}{\sqrt{ncor}} \hat{n}_{i0}^s \hat{n}_{00}^{s*} \end{bmatrix} \quad (35)$$

which then can be divided into its signal and noise components:

$$\hat{\omega}_{i0K+1} = R_{K+1} \left| \hat{W}_{i0K} \right|^2 \hat{s}_{iK} \hat{s}_{0K}^* \quad (36)$$

$$\hat{\eta}_{i0K+1} = \frac{R_{K+1}}{\sqrt{ncor}} \left| \hat{W}_{i0K} \right|^2 [\hat{s}_{iK} \hat{n}_{00}^{s*} + \hat{n}_{i0}^s \hat{s}_{0K}^* + \hat{n}_{i0}^s \hat{n}_{00}^{s*}] \quad (37)$$

In these expressions, the averaged-signal  $\hat{s}_{0K}$  is equal to  $\hat{s}_{ik}$  because  $\hat{s}_{ik}$  is constant over time and antennas.

Following the steps taken with the combined signal in the previous section, the average of the (non-normalized) weight-signal power, when the signals are relatively constant in time and the array is in lock, is

$$|\hat{\omega}_{iK+1}|^2 = R_{K+1}^2 \left| \hat{W}_{i0K} \right|^4 |\hat{s}_i|^2 |\hat{s}_0|^2 \quad (38)$$

where  $|\hat{s}_0|^2 = |\hat{s}_i|^2$  represents the signal power from each of the antennas. The average of the (non-normalized) weight-noise power, again when the noise powers are independent and relatively constant in time, is

$$|\hat{\eta}_{i0K+1}|^2 = R_{K+1}^2 \left| \hat{W}_{i0K} \right|^4 \frac{|\hat{s}_i|^2 |\hat{n}_0^s|^2 + |\hat{n}_i^s|^2 |\hat{s}_0|^2 + |\hat{n}_i^s|^2 |\hat{n}_0^s|^2}{ncor} \quad (39)$$

With factoring of terms, this becomes

$$|\hat{\eta}_{i0K+1}|^2 = R_{K+1}^2 \left| \hat{W}_{i0K} \right|^4 \frac{\left( |\hat{s}_i|^2 + |\hat{n}_i^s|^2 \right) \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) - |\hat{s}_i|^2 |\hat{s}_0|^2}{ncor} \quad (40)$$

where, again,  $|\hat{s}_0|^2 = |\hat{s}_i|^2$  and  $|\hat{n}_0^s|^2 = |\hat{n}_i^s|^2$  represent the typical signal and noise powers from each of the antennas, and are not dependent on time. Using these weight-signal and weight-noise powers, we get the following for the SIMPLE weight SNR:

$$\gamma_i^{SI} = \frac{|\hat{s}_i|^2 |\hat{s}_0|^2 ncor}{\left[ \left( |\hat{s}_i|^2 + |\hat{n}_i^s|^2 \right) \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) - |\hat{s}_i|^2 |\hat{s}_0|^2 \right]} \quad (41)$$

which, with some manipulation of terms, becomes

$$\gamma_i^{SI} = \frac{ncor}{\left[ \left(1 + |\hat{n}_i^s|^2 / |\hat{s}_i|^2\right) \left(1 + |\hat{n}_0^s|^2 / |\hat{s}_0|^2\right) - 1 \right]} \quad (42)$$

If the individual antenna SNRs are  $\alpha_i = |\hat{s}_i|^2 / |\hat{n}_i^s|^2$  and  $\alpha_0 = |\hat{s}_0|^2 / |\hat{n}_0^s|^2$ , then we get

$$\gamma_i^{SI} = \frac{ncor}{\left[ (1 + 1/\alpha_i) (1 + 1/\alpha_0) - 1 \right]} \quad (43)$$

This result exhibits, not surprisingly, that the weight SNR for SIMPLE depends on the correlation averaging time,  $ncor$ , and on the SNRs of the two antennas in the correlation pair.

## V. Calculation of the SUMPLE Weight

In this section, we calculate the weight for SUMPLE. Using expressions for signal and weight similar to those defined in Section III,

$$\hat{S}_{ik} = [\hat{s}_{ik} + \hat{n}_{ik}^s] \quad (44)$$

and

$$\hat{W}_{iK} = [\hat{\omega}_{iK} + \hat{\eta}_{iK}] \quad (45)$$

where the subscript  $i$  refers to the antenna that is correlated against the sum of the others (which acts as a kind of reference antenna), we get the following for the SUMPLE weight in the next, or  $(K + 1)$ th, time interval:

$$\hat{W}_{jK+1} = R_{K+1} \hat{W}_{jK} \left\{ \frac{1}{ncor} \sum_{k=Kncor}^{(K+1)ncor-1} \left[ \left( \hat{S}_{jk} \hat{W}_{jK}^* \right) \sum_{i=0, i \neq j}^{N_{ant}-1} \left( \hat{S}_{ik}^* \hat{W}_{iK} \right) \right] \right\} \quad (46)$$

The outer sum over  $ncor$  time points (symbol times) corresponds to the correlation averaging interval, and the inner sum over all of the antennas except the  $j$ th antenna corresponds to the “reference” antenna. The expression within the curly brackets is a correction, an incremental weight, which is combined with the original weight used in the interval to yield the weight for the next time interval (see Fig. 4 for a diagram of the various intervals). The factor  $R_{K+1}$  provides normalization to guarantee the sum of the weights-squared over the antennas is equal to the number of antennas and prevents successive calculations of the weight amplitude from becoming unstable:

$$\sum_{i=0}^{N_{ant}-1} \hat{W}_{iK} \hat{W}_{iK}^* = N_{ant} \quad (47)$$

Using the earlier definition for the combining output,  $\hat{C}_k = [\hat{c}_k + \hat{n}_k^c]$ , the weight can be re-written as

$$\hat{W}_{jK+1} = R_{K+1} \hat{W}_{jK} \frac{1}{ncor} \sum_{k=Kncor}^{(K+1)ncor-1} \left[ \hat{S}_{jk} \hat{W}_{jK}^* \left( \hat{C}_k^* - \hat{S}_{jk}^* \times \hat{W}_{jK} \right) \right] \quad (48)$$

Further expansion of this expression, using the above signal and weight definitions, yields

$$\hat{W}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 \frac{1}{ncor} \sum_{k=Kncor}^{(K+1)ncor-1} \begin{bmatrix} \hat{s}_{jk} \left( \hat{c}_k^* - \hat{s}_{jk}^* \hat{\omega}_{jK} - \hat{s}_{jk}^* \hat{\eta}_{jK} \right) \\ + \hat{s}_{jk} \left( \hat{n}_k^{c*} - \hat{n}_{jk}^{s*} \hat{\omega}_{jK} - \hat{n}_{jk}^{s*} \hat{\eta}_{jK} \right) \\ + \hat{n}_{jk}^s \left( \hat{c}_k^* - \hat{s}_{jk}^* \hat{\omega}_{jK} - \hat{s}_{jk}^* \hat{\eta}_{jK} \right) \\ + \hat{n}_{jk}^s \left( \hat{n}_k^{c*} - \hat{n}_{jk}^{s*} \hat{\omega}_{jK} - \hat{n}_{jk}^{s*} \hat{\eta}_{jK} \right) \end{bmatrix} \quad (49)$$

Assuming a relatively constant signal and noise power over the  $ncor$  summation (corresponding to integration, or averaging, in time) allows us to perform that sum, to give

$$\hat{W}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 \begin{bmatrix} \hat{s}_{jK} \left( \hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK} \right) \\ + \frac{1}{\sqrt{ncor}} \left[ \hat{s}_{jK} \left( \hat{n}_0^{c*} - \hat{n}_{j0}^{s*} \hat{\omega}_{jK} - \hat{n}_{j0}^{s*} \hat{\eta}_{jK} \right) \right] \\ + \frac{1}{\sqrt{ncor}} \left[ \hat{n}_{j0}^s \left( \hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK} \right) \right] \\ + \frac{1}{\sqrt{ncor}} \left[ \hat{n}_{j0}^s \left( \hat{n}_0^{c*} - \hat{n}_{j0}^{s*} \hat{\omega}_{jK} - \hat{n}_{j0}^{s*} \hat{\eta}_{jK} \right) \right] \end{bmatrix} \quad (50)$$

This expression can be divided into its signal and noise components:

$$\hat{\omega}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 \hat{s}_{jK} \left( \hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK} \right) \quad (51)$$

$$\hat{\eta}_{jK+1} = \frac{R_{K+1}}{\sqrt{ncor}} \left| \hat{W}_{jK} \right|^2 \begin{bmatrix} \left[ \hat{s}_{jK} \left( \hat{n}_0^{c*} - \hat{n}_{j0}^{s*} \hat{\omega}_{jK} - \hat{n}_{j0}^{s*} \hat{\eta}_{jK} \right) \right] + \\ \left[ \hat{n}_{j0}^s \left( \hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK} \right) \right] + \\ \left[ \hat{n}_{j0}^s \left( \hat{n}_0^{c*} - \hat{n}_{j0}^{s*} \hat{\omega}_{jK} - \hat{n}_{j0}^{s*} \hat{\eta}_{jK} \right) \right] \end{bmatrix} \quad (52)$$

These two expressions can be further reduced to

$$\hat{\omega}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 \left[ \hat{s}_{jK} \hat{c}_K^* - |\hat{s}_{jK}|^2 \left( \hat{\omega}_{jK} + \hat{\eta}_{jK} \right) \right] \quad (53)$$

$$\hat{\eta}_{jK+1} = \frac{R_{K+1}}{\sqrt{ncor}} \left| \hat{W}_{jK} \right|^2 \begin{bmatrix} \left( \hat{s}_{jK} + \hat{n}_{j0}^s \right) \left[ \hat{n}_0^{c*} - \hat{n}_{j0}^{s*} \left( \hat{\omega}_{jK} + \hat{\eta}_{jK} \right) \right] + \\ \hat{n}_{j0}^s \left[ \hat{c}_K^* - \hat{s}_{jK}^* \left( \hat{\omega}_{jK} + \hat{\eta}_{jK} \right) \right] \end{bmatrix} \quad (54)$$

In these expressions, the averaged signal  $\hat{s}_{jK}$  is equal to  $\hat{s}_{jk}$  because  $\hat{s}_{jk}$  is constant over time.

Following the steps taken with the SIMPLE weight in the last section, the average of the weight-signal power is

$$|\hat{\omega}_{jK+1}|^2 = R_{K+1}^2 \left| \hat{W}_{jK} \right|^4 |\hat{s}_0|^2 \left[ |\hat{c}_K|^2 + |\hat{s}_0|^2 - (\hat{c}_K^* \hat{s}_{jK} \hat{\omega}_{jK}^* + \hat{c}_K \hat{s}_{jK}^* \hat{\omega}_{jK}) \right] \quad (55)$$

and the average of the weight-noise power is

$$|\hat{\eta}_{jK+1}|^2 = \frac{R_{K+1}^2}{ncor} \left| \hat{W}_{jK} \right|^4 |\hat{n}_0^s|^2 \left[ \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) (N_{ant} - 1) + \left[ |\hat{c}_K|^2 + |\hat{s}_0|^2 - (\hat{c}_K^* \hat{s}_{jK} \hat{\omega}_{jK}^* + \hat{c}_K \hat{s}_{jK}^* \hat{\omega}_{jK}) \right] \right] \quad (56)$$

In each of these derivations, we have assumed the signals and noises are relatively constant in time and the antennas are all equal in size and performance, where, again,  $|\hat{s}_0|^2 = |\hat{s}_{jK}|^2$  and  $|\hat{n}_0^s|^2 = |\hat{n}_{jK}^s|^2$  represent the typical signal and noise powers from each of the antennas, and are assumed not to depend on time. Both of these expressions have been further simplified by noticing that the expression for the sum of the weight-square,

$$\sum_{j=0}^{N_{ant}-1} \hat{W}_{jK} \hat{W}_{jK}^* = N_{ant} \quad (57)$$

under the same conditions of equal antennas, signals, and noise, implies

$$\left| \hat{W}_{jK} \right|^2 = |\hat{\omega}_{iK}|^2 + |\hat{\eta}_{iK}|^2 \approx 1 \quad (58)$$

By plugging the weight powers into the weight-square sum above, it is possible to derive an expression for the normalization factor  $R_{K+1}$  as follows:

$$1 = \frac{1}{N_{ant}} \sum_{j=0}^{N_{ant}-1} \left[ |\hat{\omega}_{jK+1}|^2 + |\hat{\eta}_{jK+1}|^2 \right] \quad (59)$$

which, upon substitution, becomes

$$1 = \frac{R_{K+1}^2}{N_{ant}} \sum_{j=0}^{N_{ant}-1} \left[ \begin{aligned} &|\hat{s}_0|^2 \left[ |\hat{c}_K|^2 + |\hat{s}_0|^2 - (\hat{c}_K^* \hat{s}_{jK} \hat{\omega}_{jK}^* + \hat{c}_K \hat{s}_{jK}^* \hat{\omega}_{jK}) \right] \\ &+ \frac{1}{ncor} |\hat{s}_0|^2 |\hat{n}_0^s|^2 (N_{ant} - 1) \\ &+ \frac{1}{ncor} |\hat{n}_0^s|^2 \left[ |\hat{c}_K|^2 + |\hat{s}_0|^2 - (\hat{c}_K^* \hat{s}_{jK} \hat{\omega}_{jK}^* + \hat{c}_K \hat{s}_{jK}^* \hat{\omega}_{jK}) \right] \\ &+ \frac{1}{ncor} |\hat{n}_0^s|^2 |\hat{n}_0^s|^2 (N_{ant} - 1) \end{aligned} \right] \quad (60)$$

Note that two of the sums in this expression collapse to

$$\frac{1}{N_{ant}} \sum_{j=0}^{N_{ant}-1} \left[ (\hat{c}_K^* \hat{s}_{jK} \hat{\omega}_{jK}^* + \hat{c}_K \hat{s}_{jK}^* \hat{\omega}_{jK}) \right] = \frac{2}{N_{ant}} |\hat{c}_K|^2 \quad (61)$$

Consequently, for the normalization factor we get

$$R_{K+1}^{-2} = \begin{bmatrix} |\hat{s}_0|^2 \left[ \frac{(N_{ant}-2)}{N_{ant}} |\hat{c}_K|^2 + |\hat{s}_0|^2 \right] \\ + \frac{1}{ncor} |\hat{s}_0|^2 |\hat{n}_0^s|^2 (N_{ant} - 1) \\ + \frac{1}{ncor} |\hat{n}_0^s|^2 \left[ \frac{(N_{ant}-2)}{N_{ant}} |\hat{c}_K|^2 + |\hat{s}_0|^2 \right] \\ + \frac{1}{ncor} |\hat{n}_0^s|^4 (N_{ant} - 1) \end{bmatrix} \quad (62)$$

If we define the weight SNR as

$$\gamma_j^{SU} = \frac{|\hat{\omega}_{jK}|^2}{|\hat{\eta}_{jK}|^2} \quad (63)$$

then we can derive the following with the use of the weight-square sum constraint, Eq. (59), given above:

$$|\hat{\omega}_{jK}|^2 = \frac{\gamma_j^{SU}}{\gamma_j^{SU} + 1} \quad (64)$$

The weight-signal power then can be used in the weight-noise power to get

$$|\hat{\eta}_{jK+1}|^2 = \frac{R_{K+1}^2}{ncor} |\hat{n}_0^s|^2 \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) (N_{ant} - 1) + \frac{|\hat{\omega}_{jK+1}|^2 |\hat{n}_0^s|^2}{ncor |\hat{s}_0|^2} \quad (65)$$

This in turn can be plugged into the weight SNR expression above to get

$$\frac{|\hat{\eta}_{jK}|^2}{|\hat{\omega}_{jK}|^2} = \frac{1}{\gamma_j^{SU}} = \frac{R_K^2 (\gamma_j^{SU} + 1)}{\gamma_j^{SU} ncor} |\hat{n}_0^s|^2 \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) (N_{ant} - 1) + \frac{|\hat{n}_0^s|^2}{ncor |\hat{s}_0|^2} \quad (66)$$

Solving the above equation for  $\gamma_j^{SU}$  gives the following as the SNR:

$$\gamma_j^{SU} = \frac{\alpha ncor - R_{K+1}^2 |\hat{s}_0|^2 \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) (N_{ant} - 1)}{\left[ 1 + R_{K+1}^2 |\hat{s}_0|^2 \left( |\hat{s}_0|^2 + |\hat{n}_0^s|^2 \right) (N_{ant} - 1) \right]} \quad (67)$$

where  $\alpha = |\hat{s}_0|^2 / |\hat{n}_0^s|^2$  also has been used. Setting  $\mu = R_{K+1} |\hat{s}_0|^2 (N_{ant} - 1) \approx 1$ , we get

$$\gamma_j^{SU} = \frac{\alpha ncor - \mu^2 (1 + 1/\alpha) / (N_{ant} - 1)}{\left[ 1 + \mu^2 (1 + 1/\alpha) / (N_{ant} - 1) \right]} \quad (68)$$

Further rearranging of terms yields

$$\gamma_j^{SU} = \frac{ncor - (1/\alpha) (1 + 1/\alpha) / \left( \frac{N_{ant} - 1}{\mu^2} \right)}{(1 + 1/\alpha) \left( 1 + 1 / \left( \left( \frac{N_{ant} - 1}{\mu^2} \right) \alpha \right) \right) - 1} \quad (69)$$

This can be expressed as

$$\gamma_j^{SU} = \gamma_0^{SU} - \Delta\gamma_j^{SU} \quad (70)$$

where

$$\gamma_0^{SU} = \frac{ncor}{(1 + 1/\alpha) \left( 1 + 1/\left( \left( \frac{N_{ant} - 1}{\mu^2} \right) \alpha \right) \right) - 1} \quad (71)$$

is the principal component, and

$$\Delta\gamma_j^{SU} = \frac{(1 + 1/\alpha) / \left( \frac{N_{ant} - 1}{\mu^2} \right)}{1 + (1 + 1/\alpha) / \left( \frac{N_{ant} - 1}{\mu^2} \right)} \quad (72)$$

is a correction to the principal component.

The above results demonstrate that the weight SNR for SUMPLE, just like SIMPLE, depends on the correlation averaging time,  $ncor$ , and on the SNRs of the antennas that make up the correlation pairs, but also on the factor  $(N_{ant} - 1)/\mu^2$ . The principal component,  $\gamma_0^{SU}$  in Eq. (71), in the SUMPLE expression for  $\gamma_j^{SU}$  has a form just like the SIMPLE expression in Eq. (43), but with a reference antenna SNR that is larger by this factor.

## VI. Special Concerns

There are several issues with SUMPLE that must be addressed before it can be confidently used to control array antenna alignment. The first is a question of phase wandering. In SUMPLE, all the antennas are aligned to a kind of “floating” reference. Will the phase of this reference wander as a function of time, introducing unknown phase instability? A second, related, issue is the question of phase effects due to the troposphere. As the troposphere over the various antennas changes with time, what effect will this have on the phase of the “reference”? A third is the question of convergence. How will SUMPLE perform at the start of a pass, before coherence is achieved among the antennas, and how long will it take to converge?

### A. Phase Wandering

Combining is represented by the expression

$$\hat{C}_k = \sum_{i=0}^{N_{ant}-1} \hat{S}_{ik} \hat{W}_{iK}^* \quad (73)$$

The phase of  $\hat{C}_k$ , as a function of time ( $k$  in symbols), is influenced by any phase variation present in the set of weights,  $\hat{W}_{iK}$ , as well as any variation in the original signal or impressed upon that signal by the media or by the instrument. In the SIMPLE algorithm, since the weights are obtained by cross-correlation of each antenna with a single reference antenna, these weights are all locked to that reference antenna. Thus, the phase of the array will track the phase of the reference. In SUMPLE, however, the weights are

obtained by cross-correlation of each antenna with the sum of the other antennas. This sum serves as a pseudo-reference antenna, and the phase of the array will track the phase of this “reference.” Because this reference is not tied to any particular antenna, its phase error will accumulate and result in some wandering of the array phase.

We can estimate the extent of this wandering, and a remedy, by tracking the phase of the combined output of an array. Since the weight for SUMPLE is, by definition, constant over a correlation averaging time, we can average the combined output over the same interval to obtain a less noisy signal to investigate:

$$\hat{C}_{K+1} = \frac{1}{ncor} \sum_{k=(K+1)ncor}^{(K+2)ncor-1} \left[ \sum_{j=0}^{N_{ant}-1} \hat{S}_{jk} \hat{W}_{jK+1}^* \right] \quad (74)$$

This expression can be expanded to

$$\hat{C}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ \frac{1}{ncor} \sum_{k=(K+1)ncor}^{(K+2)ncor-1} (\hat{s}_{jk} \hat{\omega}_{jK+1}^* + \hat{s}_{jk} \hat{\eta}_{jK+1}^* + \hat{n}_{jk}^s \hat{\omega}_{jK+1}^* + \hat{n}_{jk}^s \hat{\eta}_{jK+1}^*) \right] \quad (75)$$

Assuming only slowly varying signal and noise power, we can perform the average over  $ncor$ :

$$\hat{C}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ \hat{s}_{jK} \hat{\omega}_{jK+1}^* + \hat{s}_{jK} \hat{\eta}_{jK+1}^* + \frac{1}{\sqrt{ncor}} (\hat{n}_{j0}^s \hat{\omega}_{jK+1}^* + \hat{n}_{j0}^s \hat{\eta}_{jK+1}^*) \right] \quad (76)$$

which splits into the combined signal and combined noise,

$$\hat{c}_{K+1} = \sum_{j=0}^{N_{ant}-1} [\hat{s}_{jK} \hat{\omega}_{jK+1}^* + \hat{s}_{jK} \hat{\eta}_{jK+1}^*] \quad (77)$$

$$\hat{n}_{K+1}^c = \frac{1}{\sqrt{ncor}} \sum_{j=0}^{N_{ant}-1} [(\hat{n}_{j0}^s \hat{\omega}_{jK+1}^* + \hat{n}_{j0}^s \hat{\eta}_{jK+1}^*)] \quad (78)$$

We retain the time dependence in the signal to permit us to address the phase effects of media and instrument. Using the expression for the weight signal derived in Eq. (51) for SUMPLE,

$$\hat{\omega}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 \hat{s}_{jK} (\hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK}) \quad (79)$$

we get

$$\hat{c}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ R_{K+1} \left| \hat{W}_{jK} \right|^2 \hat{s}_{jK} \hat{s}_{jK}^* (\hat{c}_K - \hat{s}_{jK} \hat{\omega}_{jK}^* - \hat{s}_{jK} \hat{\eta}_{jK}^*) + \hat{s}_{jK} \hat{\eta}_{jK+1}^* \right] \quad (80)$$

If the antennas are equal, then the sum over antennas can be performed to yield the following for the combined signal:

$$\hat{c}_{K+1} = R_{K+1} (N_{ant} - 1) |\hat{s}_0|^2 \hat{c}_K + \sum_{j=0}^{N_{ant}-1} [\hat{s}_{jK} \hat{\eta}_{jK+1}^*] \quad (81)$$

Since the right-most term of this expression is noise added to the combining for each iteration, it will cause the phase of the combined signal to wander. (Because the amplitude is normalized by  $R_{K+1}$  each iteration, it does not vary significantly.) This phase wandering is caused by noise in the weights. In the following, we show that this wandering can be eliminated by a correction derived from the sum of the weights.

The individual antenna weight for SUMPLE, obtained by adding the appropriate noise to Eq. (79), is

$$\hat{W}_{jK+1} = R_{K+1} \left| \hat{W}_{jK} \right|^2 [\hat{s}_{jK} (\hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK})] + \hat{\eta}_{jK+1} \quad (82)$$

If this weight is multiplied by a factor consisting of its complex conjugate and its corresponding signal amplitude, and then summed over all the antennas, we get

$$\Delta \hat{W}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ \hat{W}_{jK+1} \left( |\hat{s}_{jK}| \hat{W}_{jK}^* \right) \right] \quad (83)$$

which, when incorporating the weight expression, yields

$$\Delta \hat{W}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ R_{K+1} \left| \hat{W}_{jK} \right|^2 |\hat{s}_{jK}| \hat{W}_{jK}^* \hat{s}_{jK} (\hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK}) + |\hat{s}_{jK}| \hat{W}_{jK}^* \hat{\eta}_{jK+1} \right] \quad (84)$$

Since the weight for an antenna is derived from the correlation of the antenna signal and the sum of the other antenna signals, its phase does not contain any component that is common to all the antenna signals (either due to the troposphere, to instrumentation, or to being present in the original signals). Notice, then, that the factor used above is, *within the noise*, equal to the antenna signal vector without this common phase:

$$\hat{s}'_{jK} = |\hat{s}_{jK}| \hat{W}_{jK}^* = \hat{s}_{jK}^* \hat{U}_K^C \quad (85)$$

where  $\hat{U}_K^C$  is a vector of unit amplitude and having the common phase of the antenna signals. If we use this expression in the delta weight equation, Eq. (84), we get

$$\Delta \hat{W}_{K+1} = \sum_{j=0}^{N_{ant}-1} \left[ R_{K+1} \left| \hat{W}_{jK} \right|^2 \hat{s}_{jK}^* \hat{U}_K^C \hat{s}_{jK} (\hat{c}_K^* - \hat{s}_{jK}^* \hat{\omega}_{jK} - \hat{s}_{jK}^* \hat{\eta}_{jK}) + \hat{s}_{jK}^* \hat{U}_K^C \hat{\eta}_{jK+1} \right] \quad (86)$$

which, for equal antennas, becomes

$$\Delta \hat{W}_{K+1} = \hat{U}_K^C \left[ R_{K+1} (N_{ant} - 1) |\hat{s}_0|^2 \hat{c}_K^* + \sum_{j=0}^{N_{ant}-1} [\hat{s}_{jK}^* \hat{\eta}_{jK+1}] \right] \quad (87)$$



This expression is, within a phase offset and some noise, identical to the expression for the combined signal calculated in Eq. (81) and repeated here:

$$\hat{c}_{K+1} = R_{K+1} (N_{ant} - 1) |\hat{s}_0|^2 \hat{c}_K + \sum_{j=0}^{N_{ant}-1} [\hat{s}_{jK} \hat{\eta}_{jK+1}^*] \quad (88)$$

Therefore, the phase wandering introduced by the SUMPLE algorithm can be corrected by subtracting the phase of  $\Delta \hat{W}_{K+1}$  from the phase of  $\hat{c}_{K+1}$  (this is accomplished by multiplying the combined signal by a normalized complex conjugate version of the delta weights).

Using a random noise generator in an Excel spreadsheet, we have calculated the phase of the two expressions given above, together with the corrected phase. This is plotted in Fig. 5 for an array of 100 antennas with an averaging interval,  $ncor$ , of 6,000 symbols.

Since the phase corrections are meant to bring the antenna signals into alignment with each other, performing this operation does no damage to array coherence or signal phase.

### B. Various Phase Effects

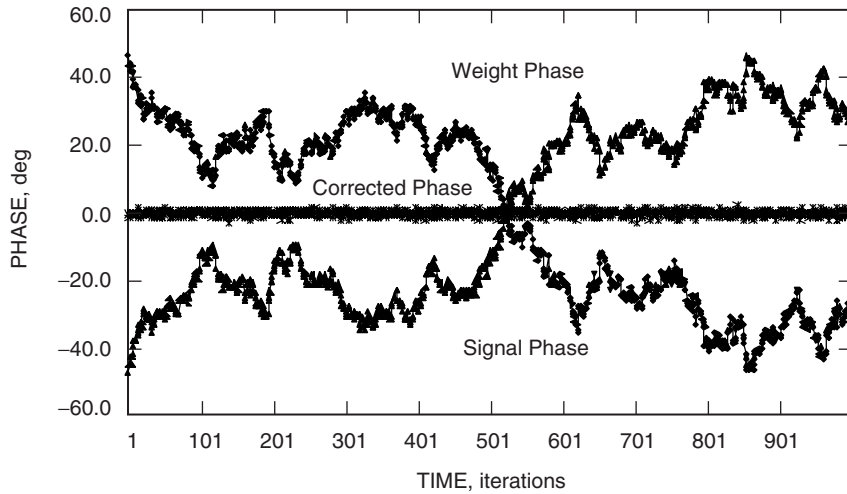
From the derivations above, the averaged combined output of an array is

$$\hat{c}_K = \sum_{j=0}^{N_{ant}-1} [\hat{s}_{jK} \hat{W}_{jK}^*] \quad (89)$$

The antenna signal in this expression can be represented by

$$\hat{s}_{jK} = |\hat{s}_0| \hat{U}_K^S \hat{U}_{jK}^G \hat{U}_{jK}^A \hat{U}_{jK}^I \quad (90)$$

where the various unit vectors contain the phase present in the incoming signal,  $\hat{U}_K^S$  (which does not vary with antenna because of the assumed large distance of the spacecraft, but may vary with time due



**Fig. 5. SUMPLE phase wander and correction versus iteration for 100 antennas. The corrected phase happens to be 0, but need not be in general.**

to Doppler and onboard oscillator instabilities); the phase errors coming from poorly known platform parameters,  $\hat{U}_{jK}^G$  (which vary both with antenna and time); the phase introduced by the troposphere,  $\hat{U}_{jK}^A$  (which does vary with the position of the antenna and with time); and the phase introduced by the instrument,  $\hat{U}_{jK}^I$  (which may vary both with antenna and with time). Putting these effects into the combining expression yields

$$\hat{c}_K = \sum_{j=0}^{N_{ant}-1} \left[ |\hat{s}_0| \hat{U}_K^S \hat{U}_{jK}^G \hat{U}_{jK}^A \hat{U}_{jK}^I \hat{W}_{jK}^* \right] \quad (91)$$

Each of the effects that vary with antenna can be expressed as the product of a non-varying and a antenna-varying component:

$$\hat{U}_{jK}^G = \hat{U}_{0K}^G \hat{\Delta}_{jK}^G \quad (92)$$

$$\hat{U}_{jK}^A = \hat{U}_{0K}^A \hat{\Delta}_{jK}^A \quad (93)$$

$$\hat{U}_{jK}^I = \hat{U}_{0K}^I \hat{\Delta}_{jK}^I \quad (94)$$

which, when inserted into the combining expression, yields

$$\hat{c}_K = |\hat{s}_0| \hat{U}_K^S \hat{U}_{0K}^G \hat{U}_{0K}^A \hat{U}_{0K}^I \sum_{j=0}^{N_{ant}-1} \left[ \hat{\Delta}_{jK}^G \hat{\Delta}_{jK}^A \hat{\Delta}_{jK}^I \hat{W}_{jK}^* \right] \quad (95)$$

The summation, by definition of the weights, to *within the noise*, adds up to  $N_{ant}$  times a unit vector with an arbitrary total phase, giving the following for the combined output:

$$\hat{c}_K = N_{ant} |\hat{s}_0| \hat{U}_K^S \hat{U}_{0K}^G \hat{U}_{0K}^A \hat{U}_{0K}^I \hat{U}_{0K}^W \quad (96)$$

None of these phase offsets contributes to wandering in the SUMPLE algorithm in the same way as noise does, as shown in the last section, because they don't contribute iteratively. However, if the absolute phase of the combined output is required, then each of these phase offsets must be calibrated. The signal part requires knowing spacecraft Doppler and oscillator phase; the geometric part can be reduced with better platform parameters, etc. The tropospheric phase can, in principle, be calibrated if the atmospheric water vapor content over each antenna can be measured. Short of this, there always will be some residual phase that will slowly vary in time, but it can be calibrated by periodically observing a quasar radio source in a direction similar to that of the spacecraft.

### C. Convergence

A heuristic description of what happens during the startup process is as follows: If the signal phases from each of the antennas begin with random values (largely due to the instrument), the sum of these signals will result in a very small array output. Nevertheless, the sum *will* form a vector with some amplitude and phase (zero amplitude is extremely unlikely.) Each of the antenna signals then is correlated with this sum to derive a weight that, when applied to the individual antennas, will tend to bring them into phase alignment with the sum vector. Consequently, in the next iteration, the sum of the antennas should be larger and the derived weights better. After a number of iterations, the array will converge to a fixed phase.

To investigate array convergence, we can perform a series of iterations in an Excel spreadsheet using two expressions that have been derived from the various quantities obtained in Sections III and V:

$$R_{K+1}^{-2} = \frac{|\hat{n}_0|^4}{ncor} [[(N_{ant} - 2) \text{SNR}_K + \alpha] (\alpha ncor + 1) + (N_{ant} - 1) (\alpha + 1)] \quad (97)$$

and

$$\text{SNR}_{K+1} = R_{K+1}^2 (N_{ant} - 1)^2 |\hat{s}_0|^4 \text{SNR}_K + \frac{\alpha [1 + R_{K+1}^2 |\hat{s}_0|^4 (N_{ant} - 1) (1 + 1/\alpha)]}{(1 + \alpha ncor)} \quad (98)$$

Figure 6 shows the results of these calculations for several values of the averaging interval,  $ncor$ , for an array of 25 antennas. Figure 7 shows the same for an array of 100 antennas.

It is clear from these results that convergence occurs typically within 10 iterations, and fewer as the averaging interval becomes longer. It should be noted, however, that this convergence interval, in symbols, is usually longer for the higher values of  $ncor$ . For example, from Fig. 7, the interval for  $ncor = 40,000$  is approximately 120,000 symbols, while the interval for  $ncor = 2,000$  is approximately 20,000 symbols. Figure 8 shows a re-plot of Fig. 6 using symbols for the time axis instead of iterations to expose this characteristic.

If a convergence time of 10 or so iterations is, for some arraying situation, unacceptably long, then it is possible to perform, quickly, several iterations on the same  $ncor$  set of data points. Convergence then can be achieved in a single iteration. To work, however, this technique requires significantly higher ( $\times 10$ ) processing speed.

## VII. Simulations

Original efforts to simulate SIMPLE and SUMPLE were made at JPL in the late 1980s independently by R. Kahn, D. N. Fort (personal communications), and the present author, with results indicating that SUMPLE did converge in a reasonable period of time even with low SNR signals, and had significantly

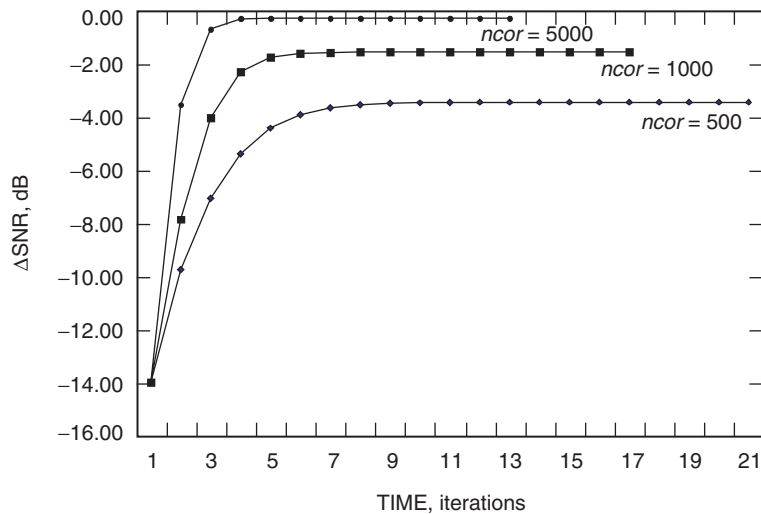
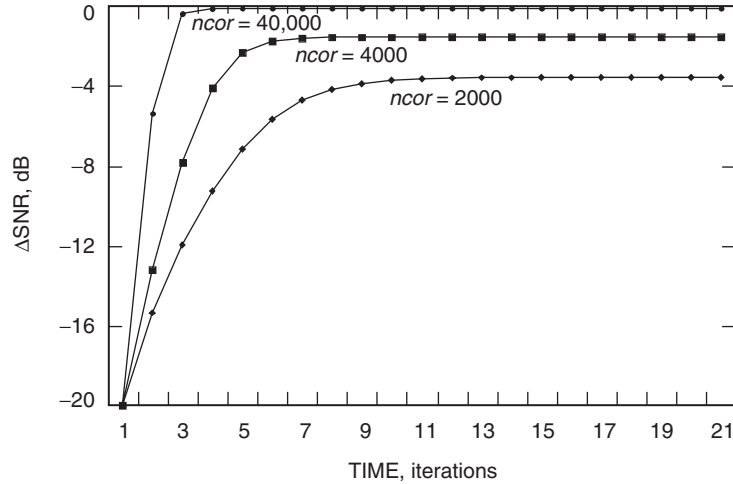
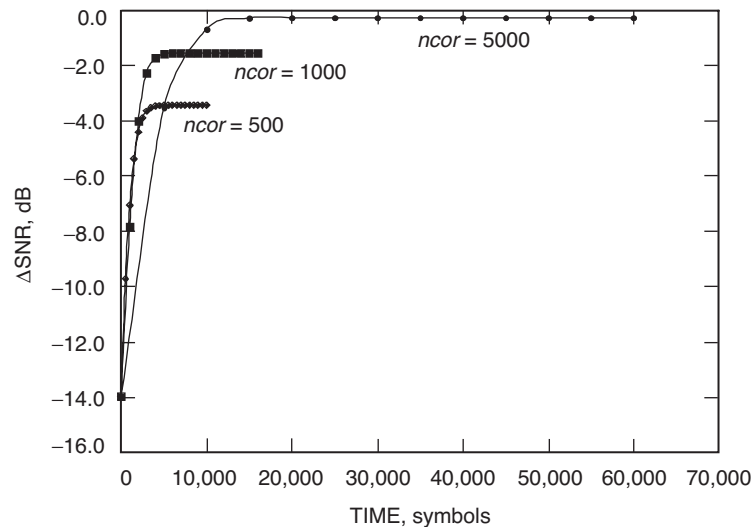


Fig. 6. Convergence times for a 25-antenna array with -5 dB output SNR.



**Fig. 7. Convergence times for a 100-antenna array with -5 dB output SNR.**



**Fig. 8. Convergence times for a 25-antenna array with -5 dB output SNR.**

improved performance over SIMPLE. Since then, additional algorithms have been developed, including LSFIT, a least-squares fitting approach by D. N. Fort,<sup>2</sup> and EIGEN, an eigen-value approach by K.-M. Cheung [9]. Fort wrote a C-language-based simulation of all four of the above algorithms. Preliminary results from this effort were presented in some of the design reviews at the time and served as a basis for proceeding with implementation of both SIMPLE and SUMPLE in the 34-meter array.<sup>3</sup>

With newer array designs being pursued in the DSN that involve up to as many as 400 antennas, the SUMPLE algorithm offered promise for providing both hardware and processing savings as well as good performance in the face of weaker signals. This promise motivated the present study, where effort has been put into both analytical analysis and more complete simulations. The software program developed by Fort served as a starting point, where improvements were made in its speed together with

<sup>2</sup> D. N. Fort, personal communication, Jet Propulsion Laboratory, Pasadena, California, 1994; see [7], p. 96.

<sup>3</sup> Ibid.

the incorporation of several features, including the ability to simulate arrays up to 400 antennas, the addition of tropospheric phase effects, self-calibration of the antenna gains, and a variety of approaches for measuring the performance of the array as it starts up and eventually achieves lock. We describe here some of the key features of this simulation program and present some typical results along with comparisons to the analytical results derived above.

## A. Approach

A full-up simulation of every detail of an antenna array may be useful at some stage of its design and implementation but is unnecessary to establish certain critical features of array performance. The purpose of the present simulation is to investigate key features of the SUMPLE algorithm for controlling the coherence of a large array and to compare this performance with a few other algorithms. The following limitations are imposed to reduce complexity and enhance speed of the simulation. None of these are believed to significantly reduce the generality or usefulness of the results.

- (1) Sampling and processing are performed at the symbol rate in a baseband channel.
- (2) No geometric effects are included, except some variation of troposphere versus antenna.
- (3) Input signal phase does not vary except due to effects of the troposphere.
- (4) Instrumental phase offsets are included, but no time variation of this phase is included.
- (5) Instrumental gain is fixed, but an option allows evaluation of gain from cross-correlation.
- (6) All generated noise is independent, but some troposphere-dependent effects are included.

The basic architecture of the program consists of a series of nested loops. The innermost is a loop over time measured in symbols, and its limit,  $ncor$ , corresponds to the number of symbols averaged to obtain the weight values used in the next averaging interval. In this loop, the signal and noise of each antenna are determined, and the various effects and weights are applied. The next loop is over time,  $T$ , measured in averaging intervals, where a weight is calculated along with a variety of performance-measuring statistics and plots. Two further outer loops surround these inner loops for the purpose of allowing a single simulation pass to vary the type of algorithm being simulated and to vary the length of  $ncor$ , the averaging interval.

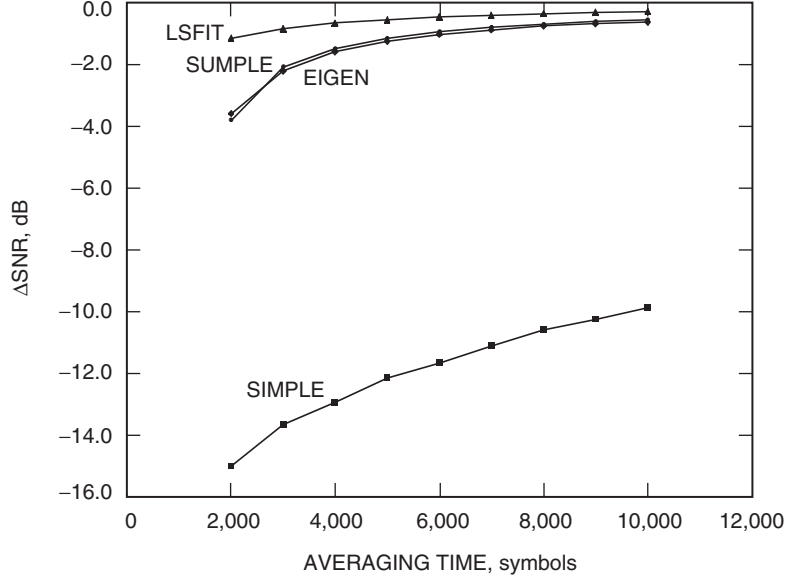
The program is run in a batch mode, with configuration parameters input via a configuration file (`com.cfg`) and through a set of execution line parameters. Output from the program is in the form of display text and plots and a log file (`com.log`). The tropospheric phase models can be input, one for each of up to 25 antennas, from files that have been separately generated. To handle simulations of more than 25 antennas, the troposphere files are repeated. The program is written in the C programming language and uses various Unix libraries together with `gnu-plot` for plots. It has been heavily commented to provide documentation. Simulation passes have been run that have taken anywhere from a few tens of seconds to a week to complete on a fast Sun workstation running the Solaris operating system.

## B. Various Comparisons

As indicated above, D. Fort performed a comparison between four different algorithms.<sup>4</sup> A repeat of this comparison has been accomplished using the present improved and extended version of the program. It appears in Fig. 2 together with a brief discussion in the second section of this article. These results are for a 25-antenna array. A similar comparison appears in Fig. 9 for a 100-antenna array.

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<sup>4</sup>Ibid.



**Fig. 9. Combining loss for four different correlation algorithms for an array of 100 antennas with output symbol SNR of  $-5$  dB.**

Figure 10 shows a comparison of the simulation results for SIMPLE and SUMPLE, with the corresponding analytical results derived above, for a 25-antenna array. Figure 11 shows the same comparison for a 100-antenna array. Note that in these two figures the points come from the simulations, and the solid lines are from the analytical expressions. It is clear from these comparisons that the analytical expressions are able to reproduce quite well what is obtained from the simulations, assuming the antennas are all of equal size and performance.

### C. Phase Wandering

With regard to the issue of phase wandering in SUMPLE, simulations show the same slow phase variations of a few degrees over time intervals of 10 to 20 iterations that were evident in the analytical expressions seen in Fig. 5. This can be seen in Fig. 12 for a configuration of 100 antennas, where the variation grows to more than 20 deg of phase in a time period of 1000 iterations at 6000 symbols per iteration.

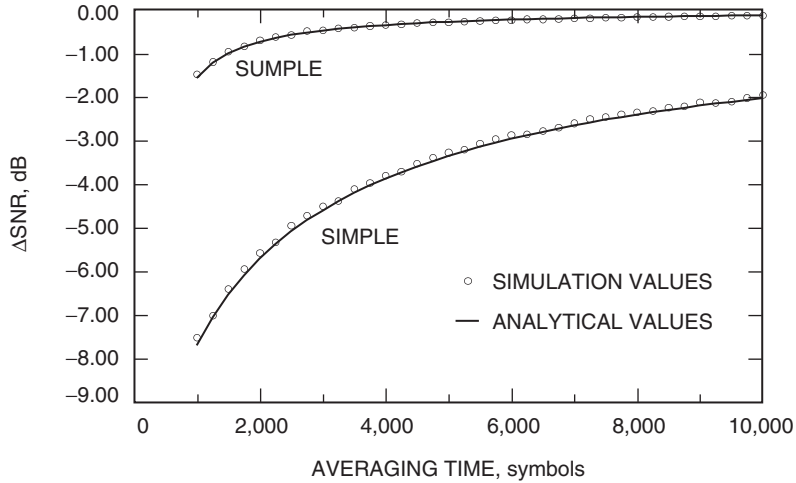
All of this wandering is eliminated, as predicted in the analysis in Section VI.A, by correcting the combined signal phase using the sum of the weights across all the antennas. The result of this also is seen in Fig. 12, where the phase is held constant to within the noise of a few degrees. Remember that the bias in the corrected phase is arbitrary.

### D. Convergence

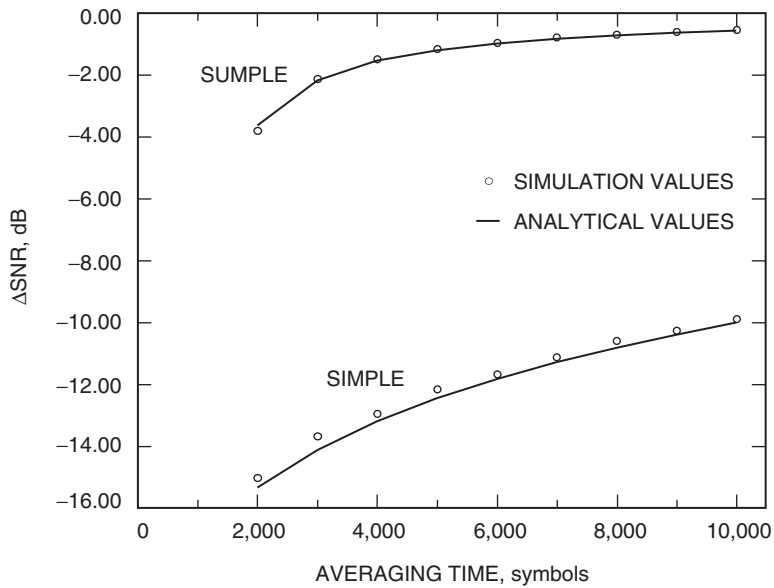
Regarding the question of array convergence, the simulations produce results similar to the analytical expressions. Figure 13 shows convergence for an array of 100 antennas, to be compared with the analytical results also plotted in Fig. 13, and in Figs. 6 and 7. Again, the simulation results are essentially the same as those derived analytically. The exact shape of the convergence in Fig. 13 will depend on the details of the noise and the beginning phase offsets, but the number of iterations to converge is still less than 10.

### E. Effects of the Troposphere

Of some interest is the response of an array to the variation of signal phase introduced as the signal passes through the troposphere on its way to the antennas. While the troposphere acts primarily as a



**Fig. 10. Comparison of combining loss for simulation and analytical results for a 25-antenna array.**



**Fig. 11. Comparison of combining loss for simulation and analytical results for a 100-antenna array.**

delay, because the signal bandwidths are narrow, the main effect is to introduce a phase shift. Tropospheric variations, both in space and time, cause the signal at each antenna to experience a different phase shift. These differences, of course, are eliminated by the process of bringing the antennas into alignment. However, an overall phase shift still remains, either due to the troposphere over the reference antenna when using SIMPLE or due to an average troposphere over the whole array when using SUMPLE. This residual phase can be eliminated only by external calibration, employing, say, water vapor radiometers (WVRs) to measure the atmospheric water content over each antenna.

To test SUMPLE in the face of tropospheric phase variation, a simulation has been performed for a 25-antenna array that includes phase offsets in the signals of each antenna due to a typical troposphere

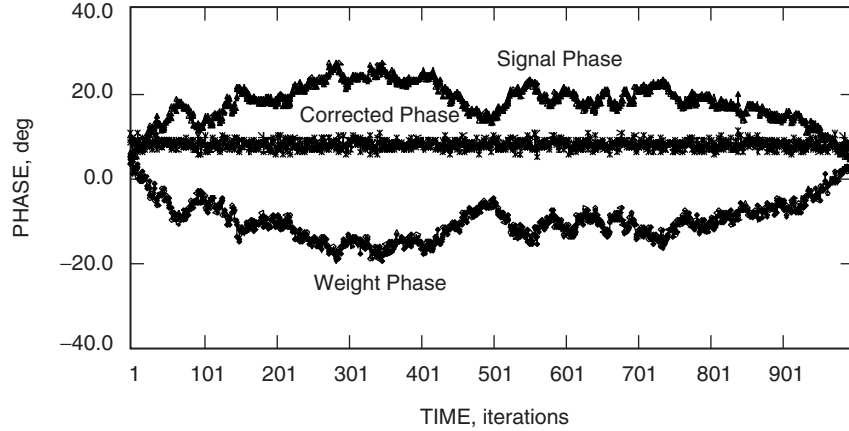


Fig. 12. Simulation of SUMPLE phase wander and correction versus iteration for 100 antennas.

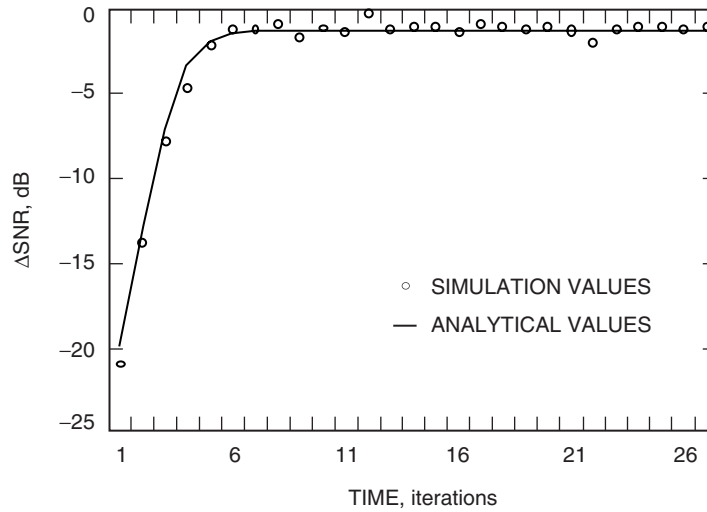


Fig. 13. Simulation of convergence versus iteration for an array of 100 antennas.

predicted from models.<sup>5</sup> To stress the SUMPLE algorithm even further, these phase offsets also were gated on and off during the course of the simulation. Figure 14 shows the tropospheric phase models for 3 of the 25 antennas. Notice that in the time intervals from 1 to 40, 81 to 120, and 161 to 200, the phase variations are due to the phase models. In the time intervals from 41 to 80 and 121 to 160, the phase variations are gated off (forced to zero).

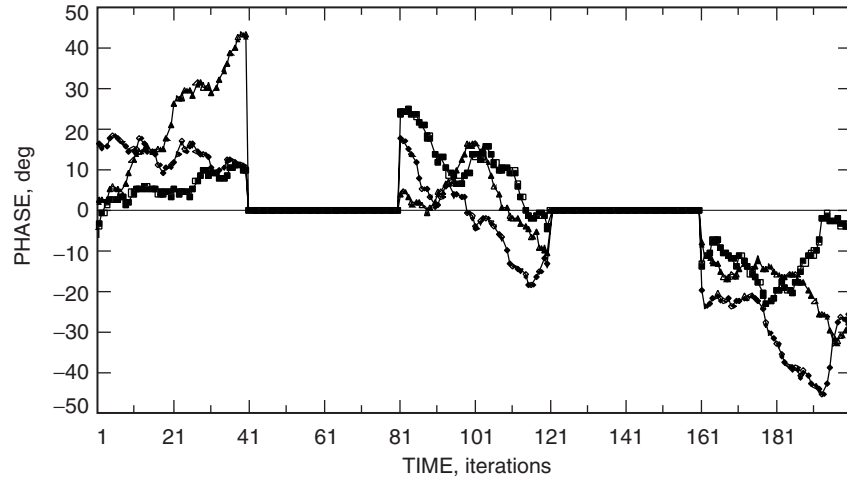
Figure 15 shows the average of the tropospheric phases for all 25 antennas, again with the gating shown in the previous figure. Figure 16 shows the combined signal phase after calibration by the average phases shown in Fig. 15. It is clear from these results that phase offsets of these types have little effect and can be calibrated, even with variations as severe as provided by the forced gating.

### F. Derivation of Antenna Gain

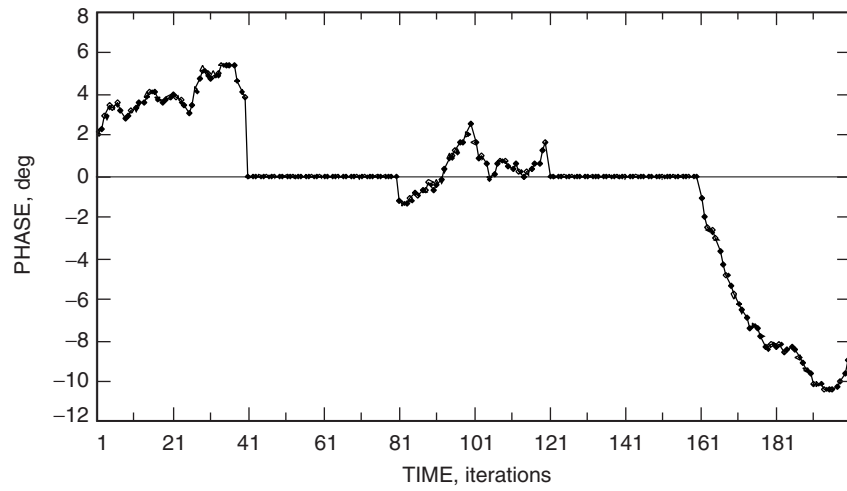
Finally, there is the question of whether the individual antenna gains can be derived from the cross-correlation amplitudes. Normally, the antenna gains are kept fixed in time and set equal to a nominal

<sup>5</sup> A. Mileant, personal communication, Jet Propulsion Laboratory, Pasadena, California, 2004.

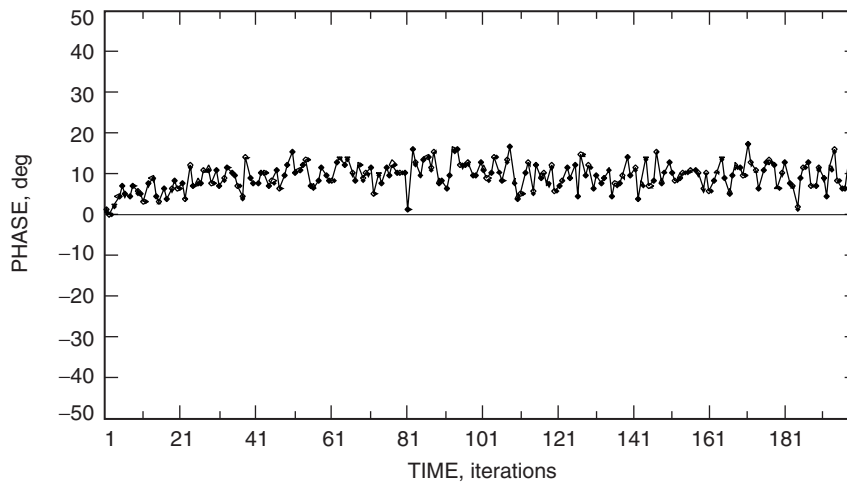




**Fig. 14. Tropospheric phase models (with gating) for 3 of 25 antennas.**



**Fig. 15. Average tropospheric phase (with gating) for all 25 antennas.**



**Fig. 16. Phase of a 25-antenna array output with varying troposphere.**

value of unity. The simulation software includes an option that evaluates the antenna gain by retaining a history of the correlation amplitudes for each antenna, and estimates this gain by averaging this history over the previous  $M$  points, where  $M$  has been set to 50 correlation-averaging intervals.

The averaging process is initialized with unity gain. Figure 17 shows one of the antenna gains as a function of time for the SIMPLE algorithm. It starts at the theoretical gain and drifts around because of the large noise in the correlations of the SIMPLE algorithm. Figure 18 shows one of the antenna gains as a function of time for the SUMPLE algorithm. Because the noise is smaller, the SUMPLE gain tends to wander less from the starting theoretical gain. A longer history average will reduce the variation but then will take longer to settle down. In any case, it is possible to obtain an estimate of the antenna gain when using either algorithm, but it is not clear if using this approach (as opposed to using a fixed estimate for each antenna) is very beneficial since it will result in a somewhat higher combining loss. Where it is probably most useful is when an antenna in a large array fails. Then the weight amplitude for that antenna will drop to zero to automatically compensate for its failure.

This can be seen in Fig. 19, where one of the antennas in a 4-antenna array drops out at time point 300 (the array size was set to 4 antennas so the effect of losing a single antenna would be more noticeable). Figure 20 shows the combining loss for this array. As expected, the loss immediately increases to about 2 dB when the antenna drops out, but then returns to a loss of about 1 dB as the weight for that antenna goes to zero. This corresponds to the difference between a 4-antenna and a 3-antenna array.

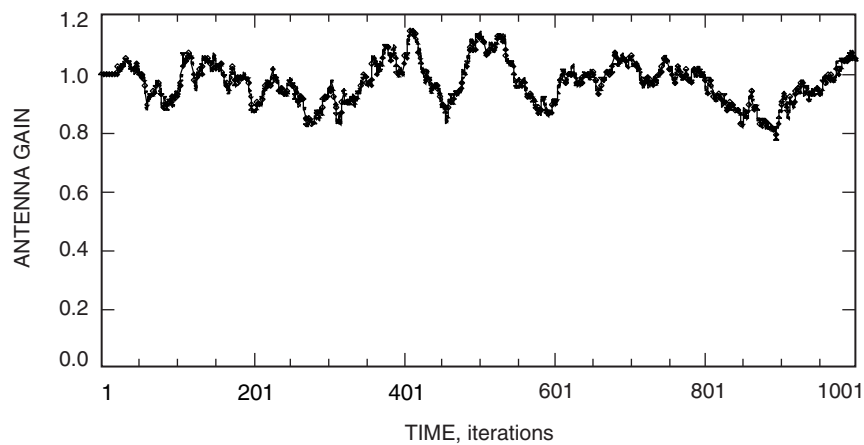


Fig. 17. Antenna gain for SIMPLE estimated from cross-correlation.

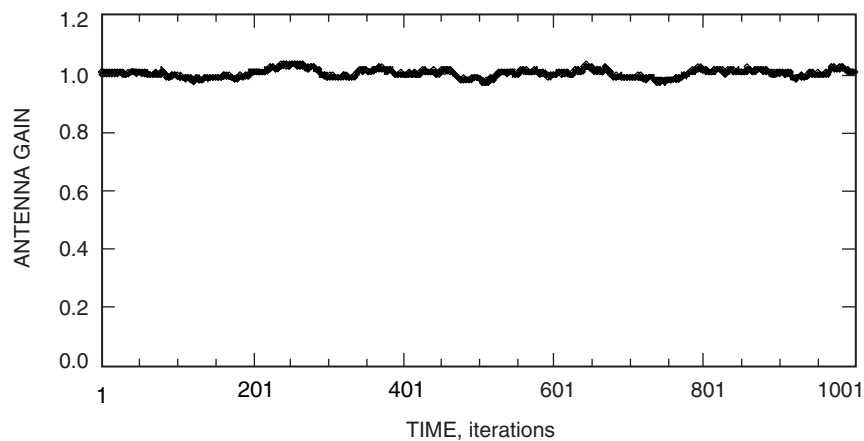
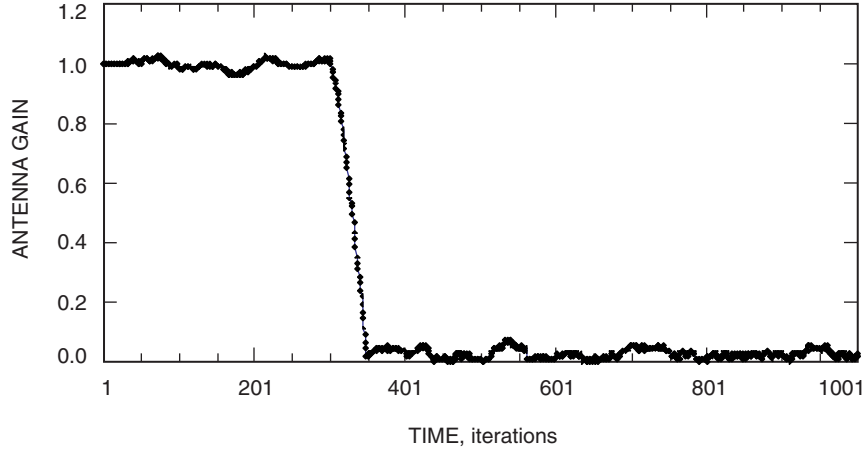
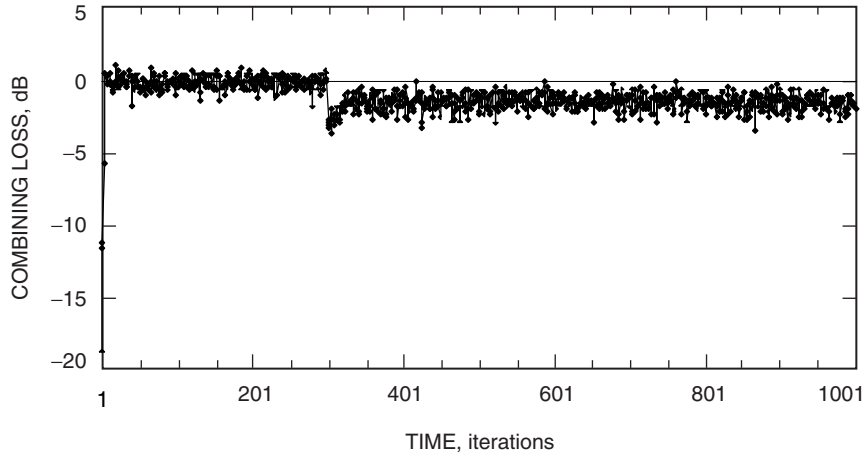


Fig. 18. Antenna gain for SUMPLE estimated from cross-correlation.



**Fig. 19. Antenna gain for SUMPLE estimated from cross-correlation as an antenna drops out.**

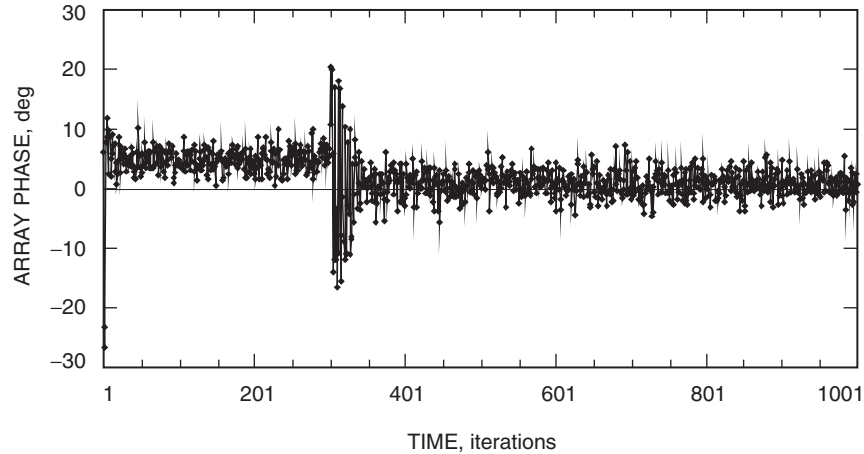


**Fig. 20. Array combining loss for SUMPLE as a single antenna drops out.**

In addition, there is an effect on the array phase when an antenna drops out. This is seen in Fig. 21, where, in addition to a larger phase scatter during the transition, a small phase shift occurs as a result of the loss of the antenna. This shift is due to the fact that the signal phase for each antenna is arbitrary before weighting is performed to bring about alignment. When using SUMPLE, the array output phase is a weighted combination of these antenna phases. If an antenna drops out, then this combination will change, resulting in a shift of the array phase. This shift will be much smaller for larger arrays.

## VIII. Conclusions

The studies presented in this article indicate a good understanding of the SIMPLE and SUMPLE algorithms, with excellent agreement between analytical derivation and simulation. We have demonstrated that the SUMPLE method of achieving array coherence performs very well in comparison to other algorithms, using less hardware and less processing power. The special concerns about SUMPLE (e.g., phase wandering) are understood and easily eliminated. Convergence of SUMPLE is robust and occurs within 10 iterations even for large arrays.



**Fig. 21. Antenna gain for SUMPLE estimated from cross-correlation as an antenna drops out.**

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